Scaling random walks on critical random trees and graphs

Kyoto University, October 2015

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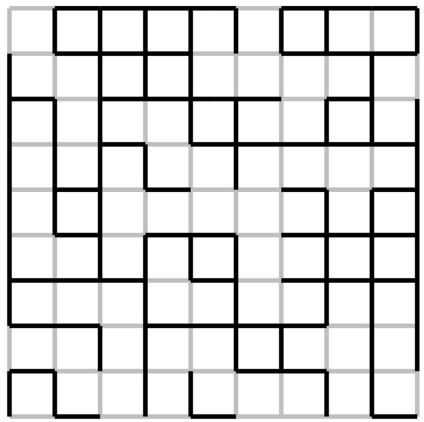
1. MOTIVATING EXAMPLES

RANDOM WALK ON PERCOLATION CLUSTERS

Bond percolation on integer lattice $\mathbb{Z}^d$ ($d \geq 2$), parameter $p > p_c$.
e.g. $p = 0.54$,



Given a configuration ω, let X^ω be the (continuous time) simple random walk on the unique infinite cluster – the 'ant in the labyrinth' [de Gennes 1976]. For $\mathbb{P}_p$-a.e. realisation of the environment,

$$q_t^\omega(x,y) = \frac{P_x^\omega(X_t^\omega = y)}{\deg_\omega(y)} \asymp c_1 t^{-d/2} e^{-c_2|x-y|^2/t}$$

for $t \geq |x-y| \vee S_x(\omega)$ [Barlow 2004].

RANDOM WALK ON PERCOLATION CLUSTERS

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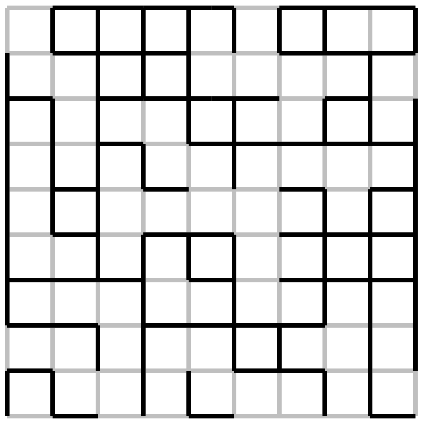


[Sidoravicius/Sznitman 2004, Biskup/Berger 2007, Mathieu/ Piatnitski 2007] For $\mathbb{P}_p$-a.e. realisation of the environment

$$\left(n^{-1} X^{\omega}_{n^2 t}\right)_{t \geq 0} \to (B_{\sigma t})_{t \geq 0}$$

in distribution, where $\sigma \in (0, \infty)$ is a deterministic constant.

ANOMALOUS BEHAVIOUR AT CRITICALITY

At criticality, $p = p_c$, physicists conjectured that the associated random walks had an anomalous **spectral dimension** [Alexander/Orbach 1982]: for every $d \geq 2$,

$$d_s = -2 \lim_{n\to\infty} \frac{\log P_x^\omega(X_{2n}^\omega = x)}{\log n} = \frac{4}{3}.$$

[Kesten 1986] constructed the law of the **incipient infinite cluster** in two dimensions, i.e.

$$\mathbb{P}_{\mathrm{IIC}} = \lim_{n\to\infty} \mathbb{P}_{p_c}\left(\cdot \middle| 0 \leftrightarrow \partial[-n,n]^2 \right),$$

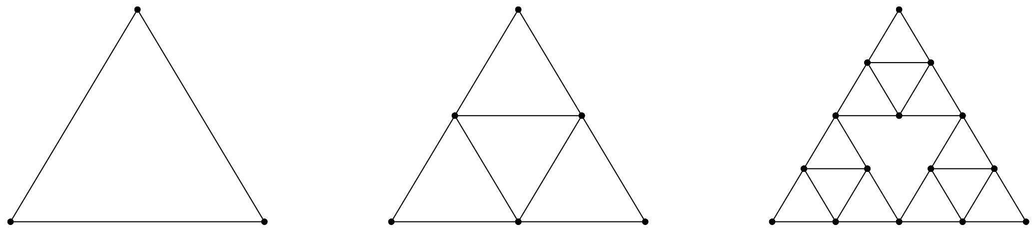
and showed that random walk on the IIC in two dimensions satisfies:

$$\left(n^{-\frac{1}{2}+\varepsilon} X_n^{\mathrm{IIC}}\right)_{n\geq 0}$$

is tight – this shows the walk is **subdiffusive**.

ANOMALOUS DIFFUSIONS ON FRACTALS

Interest from physicists [Rammal/Toulouse 1983], and construction of diffusion on fractals such as the Sierpinski gasket:



[Barlow/Perkins 1988] constructed diffusion (see also [Kigami 1989]), and established **sub-Gaussian** heat kernel bounds:

$$q_t(x,y) \asymp c_1 t^{-d_s/2} \exp\left\{-c_2(|x-y|^{d_w}/t)^{\frac{1}{d_w-1}}\right\}.$$

NB. $d_s/2 = d_f/d_w$ – the **Einstein relation**. More robust techniques applicable to random graphs since developed.

THE '$d=\infty$' CASE

Let T be a d-regular tree. Then $p_c = 1/d$. We can define

$$\mathbb{P}_{\mathrm{IIC}} = \lim_{n\to\infty} \mathbb{P}_{p_c}\left(\,\cdot\,|\,\rho \leftrightarrow \text{generation } n\right),$$

e.g. [Kesten 1986].

[Barlow/Kumagai 2006] show AO conjecture holds for $\mathbb{P}_{\mathrm{IIC}}$-a.e. environment, $\mathbb{P}_{\mathrm{IIC}}$-a.s. subdiffusivity

$$\lim_{n\to\infty} \frac{\log E_\rho^{\mathrm{IIC}}(\tau_n)}{\log n} = 3,$$

and sub-Gaussian annealed heat kernel bounds.

Similar techniques used/results established for oriented percolation in high dimensions [Barlow/Jarai/Kumagai/Slade 2008], invasion percolation on a regular tree [Angel/Goodman/den Hollander/Slade 2008], see also [Kumagai/Misumi 2008].

PROGRESS IN HIGH DIMENSIONS

Law $\mathbb{P}_{\mathrm{IIC}}$ of the **incipient infinite cluster** in high dimensions constructed in [van der Hofstad/Járai 2004].

Fractal dimension (in intrinsic metric) is 2. Unique backbone, scaling limit is Brownian motion. Scaling limit of IIC is related to **integrated super-Brownian excursion** [Kozma/Nachmias 2009, Heydenreich/van der Hofstad/Hulshof/Miermont 2013, Hara/Slade 2000].

Random walk on IIC satisfies AO conjecture ($d_s = 4/3$), and behaves subdiffusively [Kozma/Nachmias 2009], e.g. $\mathbb{P}_{\mathrm{IIC}}$-a.s.,

$$\lim_{n\to\infty} \frac{\log E_0^{\omega}(\tau_n)}{\log n} = 3.$$

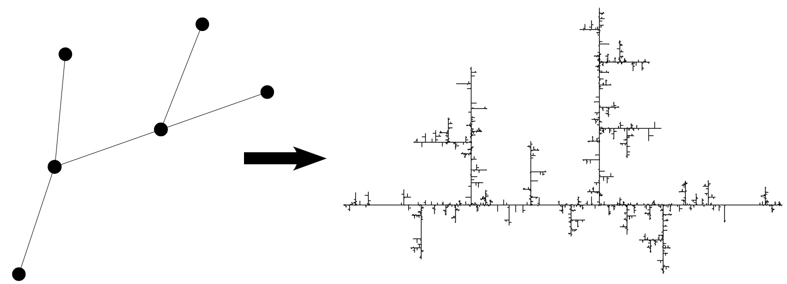
See also [Heydenreich/van der Hofstad/Hulshof 2014].

CRITICAL GALTON-WATSON TREES

Let T_n be a Galton-Watson tree with a critical (mean 1), aperiodic, finite variance offspring distribution, conditioned to have n vertices, then

$$n^{-1/2}T_n \to \mathcal{T},$$

where $\mathcal{T}$ is (up to a deterministic constant) the **Brownian continuum random tree (CRT)** [Aldous 1993], also [Duquesne/Le Gall 2002].



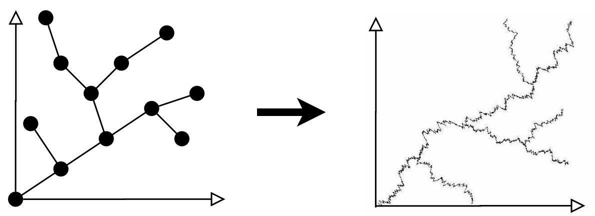
Result includes various combinatorial random trees. Similar results for infinite variance case.

CRITICAL BRANCHING RANDOM WALK

Given a graph tree T with root ρ, let $(\delta(e))_{e \in E(T)}$ be a collection of edge-indexed, i.i.d. random variables. We can use this to embed the vertices of T into $\mathbb{R}^d$ by:

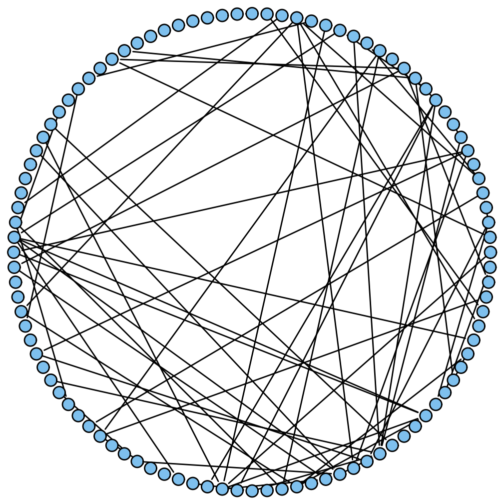
$$v \mapsto \sum_{e \in [[\rho, v]]} \delta(e).$$

If T_n are critical Galton-Watson trees with finite exponential moment offspring distribution, and $\delta(e)$ are centred and satisfy $\mathbb{P}(\delta(e) > x) = o(x^{-4})$, then the corresponding **branching random walk** has an integrated super-Brownian excursion scaling limit [Janson/Marckert 2005].



CRITICAL ERDŐS-RÉNYI RANDOM GRAPH

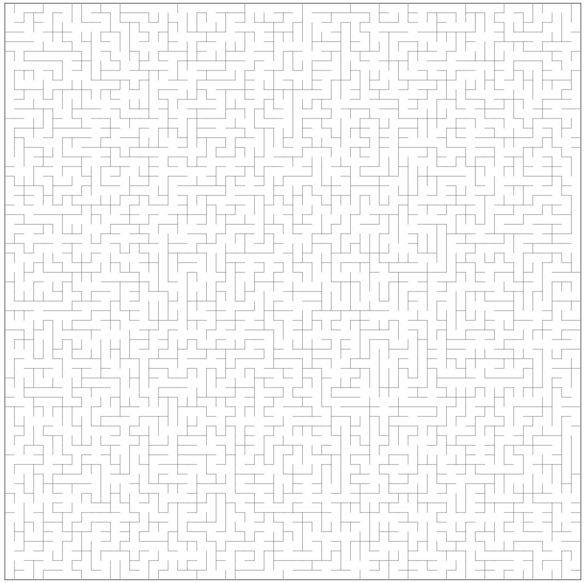
$G(n,p)$ is obtained via bond percolation with parameter p on the complete graph with n vertices. We concentrate on critical window: $p = n^{-1} + \lambda n^{-4/3}$. e.g. $n = 100$, $p = 0.01$:



All components have:
- size $\Theta(n^{2/3})$ and surplus $\Theta(1)$ [Erdős/Rényi 1960], [Aldous 1997],
- diameter $\Theta(n^{1/3})$ [Nachmias/Peres 2008].

Moreover, asymptotic structure of components is related to the Brownian CRT [Addario-Berry/Broutin/Goldschmidt 2010].

TWO-DIMENSIONAL UNIFORM SPANNING TREE



Let $\Lambda_n := [-n, n]^2 \cap \mathbb{Z}^2$.

A subgraph of the lattice is a **spanning tree** of Λ_n if it connects all vertices and has no cycles.

Let $\mathcal{U}^{(n)}$ be a spanning tree of Λ_n selected uniformly at random from all possibilities.

The UST on $\mathbb{Z}^2$, $\mathcal{U}$, is then the local limit of $\mathcal{U}^{(n)}$.

Almost-surely, $\mathcal{U}$ is a spanning tree of $\mathbb{Z}^2$. (Forest for $d > 4$.) Fractal dimension 8/5. SLE-related scaling limit.

[Aldous, Barlow, Benjamini, Broder, Häggström, Kirchoff, Lawler, Lyons, Masson, Pemantle, Peres, Schramm, Werner, Wilson,. . .]

RANDOM WALKS ON RANDOM TREES AND GRAPHS AT CRITICALITY

In the following, the aim is to:

• Introduce techniques for showing random walks on (some of) the above random graphs converge to a diffusion on a fractal;

• Study the properties of these scaling limits.

Brief outline:

2. Gromov-Hausdorff and related topologies
3. Dirichlet forms and diffusions on real trees
4. Traces and time change
5. Scaling random walks on graph trees

...

6. Fusing and the critical random graph
7. Spatial embeddings
8. Local times and cover times

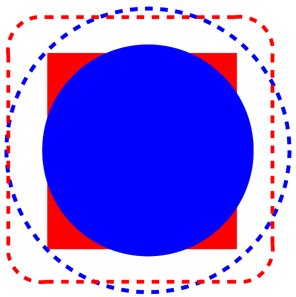
2. GROMOV-HAUSDORFF AND RELATED TOPOLOGIES

HAUSDORFF DISTANCE

The **Hausdorff distance** between two non-empty compact subsets K and K' of a metric space (M, d_M) is defined by

$$\begin{aligned} d_M^H(K, K') &:= \max\left\{\sup_{x\in K} d_M(x, K'), \sup_{x'\in K'} d_M(x', K)\right\} \\ &= \inf\left\{\varepsilon > 0 :\ K \subseteq K'_\varepsilon,\ K' \subseteq K_\varepsilon\right\}, \end{aligned}$$

where $K_\varepsilon := \{x \in M :\ d_M(x, K) \le \varepsilon\}$.



If (M, d_M) is complete (resp. compact), then so is the collection of non-empty compact subsets equipped with this metric.

GROMOV-HAUSDORFF DISTANCE

For two non-empty compact metric spaces (K, d_K), $(K', d_{K'})$, the Gromov-Hausdorff distance between them is defined by setting

$$d_{GH}(K, K') := \inf d_M^H(\phi(K), \phi'(K')),$$

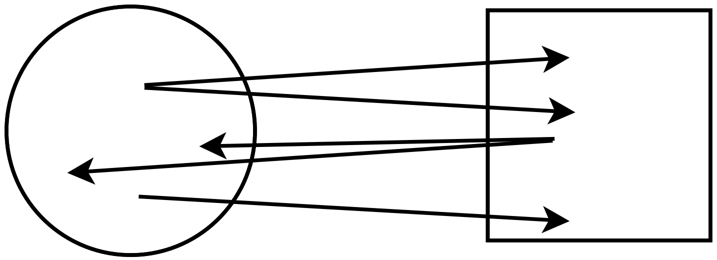
where the infimum is taken over all metric space (M, d_M) and isometric embeddings $\phi : K \to M$, $\phi' : K' \to M$.

The function d_{GH} is a metric on the collection of (isometry classes of) non-empty compact metric spaces. Moreover, the resulting metric space is complete and separable.

For background, see [Gromov 2006, Burago/Burago/Ivanov 2001].

CORRESPONDENCES

A correspondence $\mathcal{C}$ is a subset of $K \times K'$ such that for every $x \in K$ there exists an $x' \in K'$ such that $(x, x') \in \mathcal{C}$, and vice versa.



The distortion of a correspondence is:

$$\operatorname{dis} \mathcal{C} = \sup \left\{ |d_K(x,y) - d_{K'}(x',y')| : \ (x,x'),(y,y') \in \mathcal{C} \right\}.$$

An alternative characterisation of the Gromov-Hausdorff distance is then:

$$d_{GH}(K,K') = \frac{1}{2} \inf \operatorname{dis} \mathcal{C}.$$

EXAMPLE: CONVERGENCE OF GW TREES

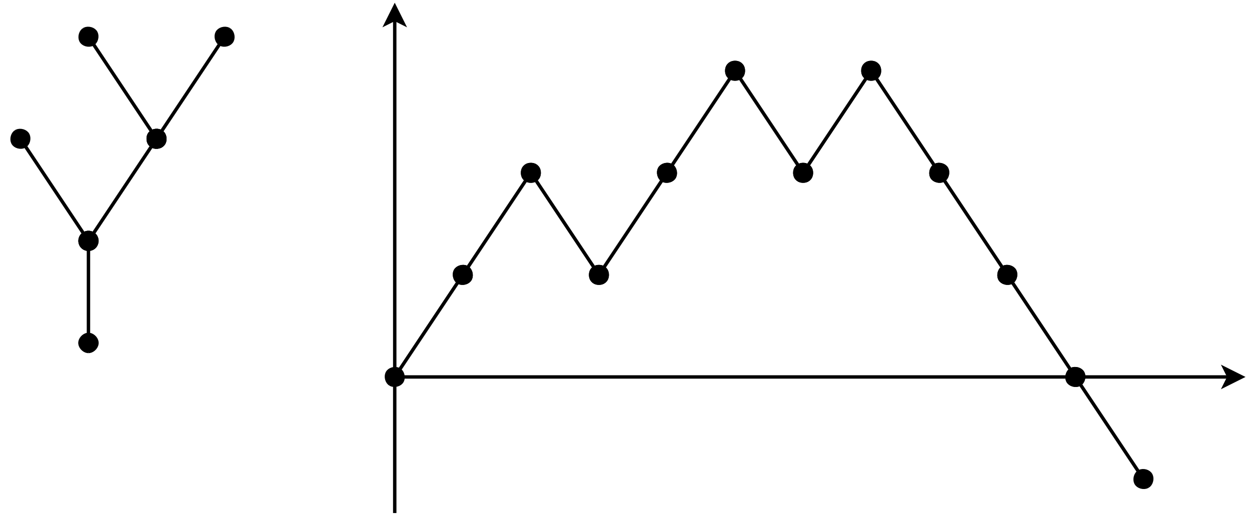
Let T_n be a Galton-Watson tree with a critical (mean 1), aperiodic, finite variance σ^2 offspring distribution, conditioned to have n vertices, then

$$\left(T_n, \frac{\sigma}{2n^{1/2}} d_{T_n}\right) \to (\mathcal{T}, d_{\mathcal{T}})$$

in distribution, with respect to the Gromov-Hausdorff topology. The limiting tree is the Brownian continuum random tree, cf. [Aldous 1993].

DISCRETE CONTOUR FUNCTION

Given an ordered graph tree T, its contour function measures the height of a particle that traces the 'contour' of the tree at unit speed from left to right.



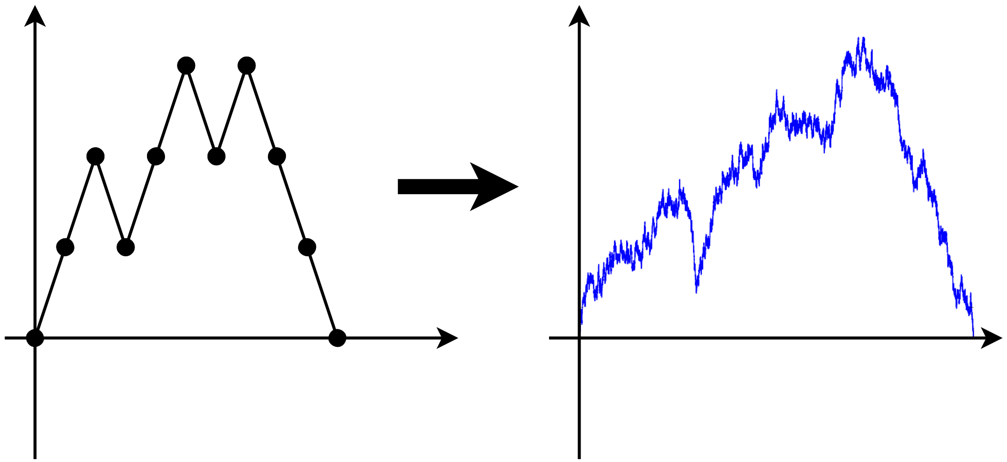
e.g. If a GW tree has a geometric, parameter $\frac{1}{2}$, distribution, then the contour function is precisely a random walk stopped at the first time it hits -1 [Harris 1952]. Conditioning tree to have n vertices equivalent to conditioning the walk to hit -1 at time $2n-1$.

CONVERGENCE OF CONTOUR FUNCTIONS

Let $(C_n(t))_{t\in[0,2n-1]}$ be the contour function of T_n. Then

$$\left(\frac{\sigma}{2n^{1/2}} C_{2(n-1)t}\right)_{t\in[0,1]} \to (B_t)_{t\in[0,1]},$$

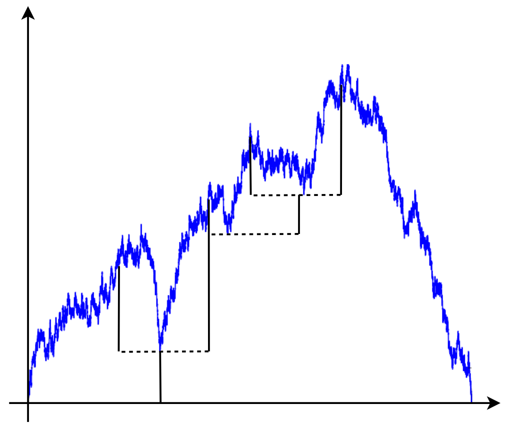
in distribution in the space $C([0,1],\mathbb{R})$, where the limit process is Brownian excursion normalised to have length one.



See [Marckert/Mokkadem 2003] for a nice general proof.

EXCURSIONS AND REAL TREES

Consider an excursion $(e(t))_{t\in[0,1]}$ – that is, a continuous function that satisfies $e(0) = e(1) = 0$ and is strictly positive for $t \in (0,1)$.



Define a distance on [0,1] by setting

$$d_e(s,t) := e(s) + e(t) - 2 \min_{r\in[s\wedge t, s\vee t]} e(r).$$

Then we obtain a (compact) real tree (see definition below) by setting $\mathcal{T}_e = [0,1]/\sim$, where $s \sim t$ iff $d_e(s,t) = 0$. [Duquesne/Le Gall 2004]

CONVERGENCE IN GH TOPOLOGY

Let $\mathcal{T} = \mathcal{T}_B$ – this is the Brownian continuum random tree.

Since $C([0,1],\mathbb{R})$ is separable, we can couple (rescaled) contour processes so that they converge almost-surely. Consider correspondence between T_n and $\mathcal{T}$ given by

$$\mathcal{C} = \{([\lceil 2(n-1)t\rceil]_n, [t]) : t \in [0,1]\},$$

where $[t]$ is the equivalence class of t with respect to $\sim$, and similarly for $[t]_n$. This satisfies

$$\operatorname{dis}\mathcal{C} \leq 4 \left\| \frac{\sigma}{2n^{1/2}} C_{2(n-1)\cdot} - B \right\|_\infty \to 0.$$

Hence

$$d_{GH}\left(\left(T_n, \frac{\sigma}{2n^{1/2}} d_{T_n}\right), (\mathcal{T}, d_{\mathcal{T}})\right) \leq 2 \left\| \frac{\sigma}{2n^{1/2}} C_{2(n-1)\cdot} - B \right\|_\infty \to 0.$$

INCORPORATING POINTS AND MEASURE

For two non-empty compact pointed metric probability measure spaces (K, d_K, μ_K, ρ_K), $(K', d_{K'}, \mu_{K'}, \rho_{K'})$, we define a distance by setting $d_{GHP}(K, K')$ to be equal to

$$\inf\left\{d_M(\phi(\rho_K), \phi'(\rho_{K'})) + d_M^H(\phi(K), \phi'(K')) + d_M^P(\mu_K \circ \phi^{-1}, \mu_{K'} \circ \phi'^{-1})\right\},$$

where the infimum is taken over all metric space (M, d_M) and isometric embeddings $\phi : K \to M$, $\phi' : K' \to M$. Here d_M^P is the Prohorov metric between probability measures on M, i.e.

$$d_M^P(\mu, \nu) = \inf\{\varepsilon : \mu(A) \leq \nu(A_\varepsilon) + \varepsilon,\ \nu(A) \leq \mu(A_\varepsilon) + \varepsilon,\ \forall A\}.$$

The function d_{GHP} is a metric on the collection of (measure and root preserving isometry classes of) non-empty compact pointed metric probability measure spaces. (Again, complete and separable.) [Abraham/Delmas/Hoscheit 2013]

EXAMPLE: GHP CONVERGENCE OF GW TREES

Let T_n be a Galton-Watson tree with a critical (mean 1), aperiodic, finite variance σ^2 offspring distribution, conditioned to have n vertices. Let μ_{T_n} be the uniform probability measure on T_n, and ρ_{T_n} its root. Then

$$\left(T_n, \frac{\sigma}{2n^{1/2}} d_{T_n}, \frac{1}{n}\mu_{T_n}, \rho_{T_n}\right) \to (\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \rho_{\mathcal{T}})$$

in distribution, with respect to the topology induced by d_{GHP}. The limiting tree is the Brownian continuum random tree. In the excursion construction $\rho_{\mathcal{T}} = [0]$, and

$$\mu_{\mathcal{T}} = \lambda \circ p^{-1},$$

where λ is Lebesgue measure on $[0,1]$ and $p : t \mapsto [t]$ is the canonical projection.

PROOF IDEA

Consider two length one excursions e and f. As before, define a correspondence $\mathcal{C} = \{([t]_e, [t]_f) : t \in [0,1]\}$, and note that $\operatorname{dis}\mathcal{C} \leq 4\|e-f\|_\infty$. Let $M = \mathcal{T}_e \sqcup \mathcal{T}_f$, with metric d_M equal to $d_{\mathcal{T}_e}$, $d_{\mathcal{T}_f}$ on $\mathcal{T}_e$, $\mathcal{T}_f$ resp., and

$$d_M(x,x') = \inf\{d_{\mathcal{T}_e}(x,y) + \tfrac{1}{2}\operatorname{dis}\mathcal{C} + d_{\mathcal{T}_f}(y',x') : (y,y') \in \mathcal{C}\},$$

for $x \in \mathcal{T}_e$, $x' \in \mathcal{T}_f$. Then

$$d_M([0]_e,[0]_f) = \tfrac{1}{2}\operatorname{dis}\mathcal{C} = d_M^H(\mathcal{T}_e,\mathcal{T}_f).$$

Moreover, if A is a measurable subset of $\mathcal{T}_e$ and $B = p_f(p_e^{-1}(A)) \subseteq \mathcal{T}_f$, then $B \subseteq A_\varepsilon$ for $\varepsilon > \frac{1}{2}\operatorname{dis}\mathcal{C}$ and

$$\mu_{\mathcal{T}_e}(A) \leq \mu_{\mathcal{T}_f}(B) \leq \mu_{\mathcal{T}_e}(A_\varepsilon).$$

By symmetry, it follows that

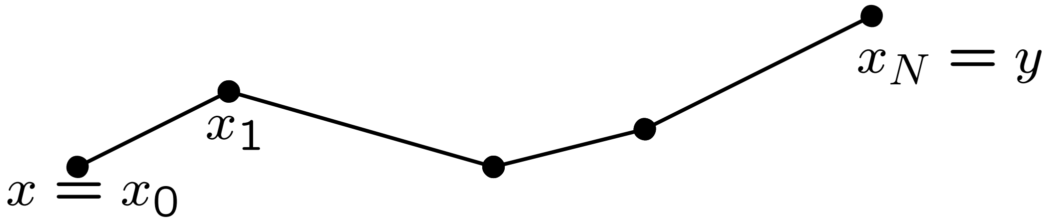
$$d_M^P(\mu_{\mathcal{T}_e},\mu_{\mathcal{T}_f}) \leq \tfrac{1}{2}\operatorname{dis}\mathcal{C}.$$

3. DIRICHLET FORMS AND DIFFUSIONS ON REAL TREES

REAL TREES

A **compact real tree** $(\mathcal{T}, d_{\mathcal{T}})$ is an arcwise-connected compact topological space containing no subset homeomorphic to the circle. Moreover, the unique arc between two points x, y is isometric to $[0, d_{\mathcal{T}}(x, y)]$. (cf. compact metric trees [Athreya/Lohr/Winter].)

In particular, the metric $d_{\mathcal{T}}$ on a real tree is additive along paths, i.e. if $x = x_0, x_1, \ldots, x_N = y$ appear in order along an arc



then

$$d_{\mathcal{T}}(x, y) = \sum_{i=1}^{N} d_{\mathcal{T}}(x_{i-1}, x_i).$$

APPROACH FOR CONSTRUCTING A DIFFUSION

Given a compact real tree $(\mathcal{T}, d_{\mathcal{T}})$ and finite Borel measure $\mu^{\mathcal{T}}$ of full support, we aim to construct a quadratic form $\mathcal{E}^{\mathcal{T}}$ that is a local, regular Dirichlet form on $L^2(\mu^{\mathcal{T}})$.

Then, through the standard association

$$\mathcal{E}^{\mathcal{T}}(f,g) = -\int_{\mathcal{T}} (\Delta_{\mathcal{T}} f) g d\mu^{\mathcal{T}} \Leftrightarrow P_t^{\mathcal{T}} = e^{t\Delta_{\mathcal{T}}},$$

define Brownian motion on $(\mathcal{T}, d_{\mathcal{T}}, \mu^{\mathcal{T}})$ to be the Markov process with generator $\Delta_{\mathcal{T}}$.

We follow the construction of [Athreya/Eckhoff/Winter 2013], see also [Krebs 1995] and [Kigami 1995].

DIRICHLET FORM DEFINITION

Let $(\mathcal{T}, d_{\mathcal{T}})$ be a compact real tree, and $\mu^{\mathcal{T}}$ be a finite Borel measure of full support. A **Dirichlet form** $(\mathcal{E}^{\mathcal{T}}, \mathcal{F}^{\mathcal{T}})$ on $L^2(\mu^{\mathcal{T}})$ is a bilinear map $\mathcal{F}^{\mathcal{T}} \times \mathcal{F}^{\mathcal{T}} \to \mathbb{R}$ that is:

- symmetric, i.e. $\mathcal{E}^{\mathcal{T}}(f,g) = \mathcal{E}^{\mathcal{T}}(g,f)$,
- non-negative, i.e. $\mathcal{E}^{\mathcal{T}}(f,f) \geq 0$,
- Markov, i.e. if $f \in \mathcal{F}^{\mathcal{T}}$, then so is $\bar{f} := (0 \vee f) \wedge 1$ and $\mathcal{E}^{\mathcal{T}}(\bar{f},\bar{f}) \leq \mathcal{E}^{\mathcal{T}}(f,f)$,
- closed, i.e. $\mathcal{F}^{\mathcal{T}}$ is complete w.r.t.
$$\mathcal{E}_1^{\mathcal{T}}(f,f) := \mathcal{E}^{\mathcal{T}}(f,f) + \int_{\mathcal{T}} f(x)^2 \mu^{\mathcal{T}}(dx),$$
- dense, i.e. $\mathcal{F}^{\mathcal{T}}$ is dense in $L^2(\mu^{\mathcal{T}})$.

It is **regular** if $\mathcal{F}^{\mathcal{T}} \cap C(\mathcal{T})$ is dense in $\mathcal{F}^{\mathcal{T}}$ w.r.t. $\mathcal{E}_1^{\mathcal{T}}$, and dense in $C(\mathcal{T})$ w.r.t. $\|\cdot\|_\infty$.

ASSOCIATION WITH SEMIGROUP

[Fukushima/Oshima/Takeda 2011, Sections 1.3-1.4] Let $(P_t^{\mathcal{T}})_{t\geq 0}$ be a strongly continuous $\mu^{\mathcal{T}}$-symmetric Markovian semigroup on $L^2(\mu^{\mathcal{T}})$. For $f \in L^2(\mu^{\mathcal{T}})$, define

$$\mathcal{E}_t^{\mathcal{T}}(f,f) := t^{-1}\int_{\mathcal{T}}(f - P_t^{\mathcal{T}} f) f d\mu^{\mathcal{T}}.$$

This is non-negative and non-decreasing in t. Let

$$\mathcal{E}^{\mathcal{T}}(f,f) := \lim_{t\downarrow 0} \mathcal{E}_t^{\mathcal{T}}(f,f), \qquad \mathcal{F}^{\mathcal{T}} := \left\{ f \in L^2(\mu^{\mathcal{T}}) : \lim_{t\downarrow 0} \mathcal{E}_t^{\mathcal{T}}(f,f) < \infty \right\}.$$

Then $(\mathcal{E}^{\mathcal{T}}, \mathcal{F}^{\mathcal{T}})$ is a Dirichlet form on $L^2(\mu^{\mathcal{T}})$. Moreover, if $\Delta_{\mathcal{T}}$ is the infinitesimal generator of $(P_t^{\mathcal{T}})_{t\geq 0}$, then $\mathcal{D}(\Delta_{\mathcal{T}}) \subseteq \mathcal{F}^{\mathcal{T}}$, $\mathcal{D}(\Delta_{\mathcal{T}})$ is dense in $L^2(\mu^{\mathcal{T}})$ and

$$\mathcal{E}^{\mathcal{T}}(f,g) = -\int_{\mathcal{T}} (\Delta_{\mathcal{T}} f) g d\mu^{\mathcal{T}}, \qquad \forall f \in \mathcal{D}(\Delta_{\mathcal{T}}),\ g \in \mathcal{F}^{\mathcal{T}}.$$

Conversely, if $(\mathcal{E}^{\mathcal{T}}, \mathcal{F}^{\mathcal{T}})$ is a Dirichlet form on $L^2(\mu^{\mathcal{T}})$, then there exists a strongly continuous $\mu^{\mathcal{T}}$-symmetric Markovian semigroup on $L^2(\mu^{\mathcal{T}})$ whose generator satisfies the above.

DIRICHLET FORMS ON GRAPHS

Let $G=(V(G),E(G))$ be a finite graph. Let $\lambda^G=(\lambda_e^G)_{e\in E(G)}$ be a collection of edge weights, $\lambda_e^G\in(0,\infty)$.

Define a quadratic form on G by setting

$$\mathcal{E}^G(f,g)=\frac{1}{2}\sum_{x,y:x\sim y}\lambda_{xy}^G\left(f(x)-f(y)\right)\left(g(x)-g(y)\right).$$

Note that, for any finite measure μ^G on $V(G)$ (of full support), $\mathcal{E}^G$ is a Dirichlet form on $L^2(\mu^G)$, and

$$\mathcal{E}^G(f,g)=-\sum_{x\in V(G)}(\Delta_G f)(x)g(x)\mu^G(\{x\}),$$

where

$$(\Delta_G f)(x):=\frac{1}{\mu^G(\{x\})}\sum_{y:\,y\sim x}\lambda_{xy}^G(f(y)-f(x)).$$

A FIRST EXAMPLE FOR A REAL TREE

For $(\mathcal{T}, d_{\mathcal{T}}) = ([0,1], \text{Euclidean})$ and μ be a finite Borel measure of full support on $[0,1]$. Let λ be Lebesgue measure on $[0,1]$, and define

$$\mathcal{E}(f,g) = \int_0^1 f'(x)g'(x)\lambda(dx), \qquad \forall f,g \in \mathcal{F},$$

where $\mathcal{F} = \{f \in C([0,1]) : f$ is abs. cont. and $f' \in L^2(\lambda)\}$. Then $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mu)$. Note that

$$\mathcal{E}(f,g) = -\int_0^1 (\Delta f)(x)g(x)\mu(dx), \qquad \forall f \in \mathcal{D}(\Delta),\ g \in \mathcal{F},$$

where $\Delta f = \frac{d}{d\mu}\frac{df}{dx}$, and $\mathcal{D}(\Delta)$ contains those f such that: f' exists and df' is abs. cont. w.r.t. μ, $\Delta f \in L^2(\mu)$, and $f'(0) = f'(1) = 0$.

If $\mu = \lambda$, then the Markov process naturally associated with Δ is reflected Brownian motion on $[0,1]$.

GRADIENT ON REAL TREES

Let $(\mathcal{T}, d_{\mathcal{T}})$ be a compact real tree, with root $\rho_{\mathcal{T}}$.

Let $\lambda^{\mathcal{T}}$ be the 'length measure' on $\mathcal{T}$, and define orientation-sensitive integration with respect to $\lambda^{\mathcal{T}}$ by

$$\int_x^y g(z)\lambda^{\mathcal{T}}(dz) = \int_{b_{\mathcal{T}}(\rho_{\mathcal{T}},x,y)}^y g(z)\lambda^{\mathcal{T}}(dz) - \int_{b_{\mathcal{T}}(\rho_{\mathcal{T}},x,y)}^x g(z)\lambda^{\mathcal{T}}(dz).$$

Write

$$\mathcal{A} = \{f \in C(\mathcal{T}) :\ f \text{ is locally absolutely continuous}\}.$$

Proposition. If $f \in \mathcal{A}$, then there exists a unique function $g \in L^1_{\mathrm{loc}}(\lambda^{\mathcal{T}})$ such that

$$f(y) - f(x) = \int_x^y g(z)\lambda^{\mathcal{T}}(dz).$$

We say $\nabla_{\mathcal{T}} f = g$.

DIRICHLET FORMS ON REAL TREES

Let $(\mathcal{T}, d_{\mathcal{T}}, \rho_{\mathcal{T}})$ be a compact, rooted real tree, and $\mu^{\mathcal{T}}$ a finite Borel measure on $\mathcal{T}$ with full support. Define

$$\mathcal{F}^{\mathcal{T}} := \left\{ f \in \mathcal{A} : \nabla_{\mathcal{T}} f \in L^2(\lambda^{\mathcal{T}}) \right\} \left(\subseteq L^2(\mu^{\mathcal{T}}) \right).$$

For $f, g \in \mathcal{F}^{\mathcal{T}}$, set

$$\mathcal{E}^{\mathcal{T}}(f,g) = \int_{\mathcal{T}} \nabla_{\mathcal{T}} f(x) \nabla_{\mathcal{T}} g(x) \lambda^{\mathcal{T}}(dx).$$

Proposition. $(\mathcal{E}^{\mathcal{T}}, \mathcal{F}^{\mathcal{T}})$ is a local, regular Dirichlet form on $L^2(\mu^{\mathcal{T}})$.

NB. By saying the Dirichlet form is **local**, it is meant that

$$\mathcal{E}^{\mathcal{T}}(f,g) = 0$$

whenever the support of f and g are disjoint.

BROWNIAN MOTION ON REAL TREES

Let $(\mathcal{T}, d_{\mathcal{T}}, \rho_{\mathcal{T}})$ be a compact, rooted real tree, and $\mu^{\mathcal{T}}$ a finite Borel measure on $\mathcal{T}$ with full support.

From the standard theory above, there is a non-positive self-adjoint operator $\Delta_{\mathcal{T}}$ on $L^2(\mu^{\mathcal{T}})$ with $\mathcal{D}(\Delta_{\mathcal{T}}) \subseteq \mathcal{F}^{\mathcal{T}}$ and

$$\mathcal{E}^{\mathcal{T}}(f,g) = -\int_{\mathcal{T}} (\Delta_{\mathcal{T}} f)(x) g(x) \mu^{\mathcal{T}}(dx),$$

for every $f \in \mathcal{D}(\Delta_{\mathcal{T}})$, $g \in \mathcal{F}^{\mathcal{T}}$.

We define **Brownian motion** on $(\mathcal{T}, d_{\mathcal{T}}, \mu^{\mathcal{T}})$ to be the Markov process

$$\left(\left(X_t^{\mathcal{T}} \right)_{t \geq 0}, \left(P_x^{\mathcal{T}} \right)_{x \in \mathcal{T}} \right)$$

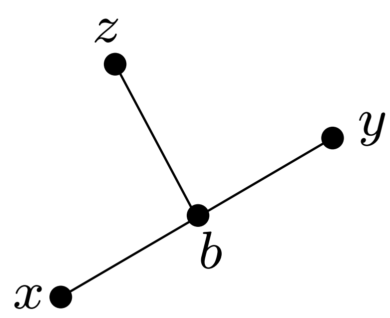
with semigroup $P_t^{\mathcal{T}} = e^{t \Delta_{\mathcal{T}}}$. Since $(\mathcal{E}^{\mathcal{T}}, \mathcal{F}^{\mathcal{T}})$ is local and regular, this is a diffusion.

PROPERTIES OF LIMITING PROCESS

Point recurrence: For $x, y \in \mathcal{T}$, $P_x^{\mathcal{T}}(\tau_y < \infty) = 1$.

Hitting probabilities: For $x, y, z \in \mathcal{T}$,

$$P_z^{\mathcal{T}}(\tau_x < \tau_y) = \frac{d_{\mathcal{T}}(b_{\mathcal{T}}(x,y,z), y)}{d_{\mathcal{T}}(x,y)}.$$



Occupation density: For $x, y \in \mathcal{T}$,

$$E_x^{\mathcal{T}} \int_0^{\tau_y} f(X_s^{\mathcal{T}}) ds = \int_{\mathcal{T}} f(x) d_{\mathcal{T}}(b_{\mathcal{T}}(x,y,z), y) \mu^{\mathcal{T}}(dz).$$

[cf. Aldous 1991]

RESISTANCE CHARACTERISATION: GRAPHS

As above, let $G = (V(G), E(G))$ be a finite graph, with edge weights $\lambda^G = (\lambda_e^G)_{e \in E(G)}$.

Suppose we view G as an electrical network with edges assigned conductances according to λ^G. Then the electrical resistance between x and y is given by

$$R_G(x,y)^{-1} = \inf\left\{\mathcal{E}^G(f,f) : \, f(x) = 1, f(y) = 0\right\}.$$

R_G is a metric on $V(G)$, e.g. [Tetali 1991], and characterises the weights (and therefore the Dirichlet form) uniquely [Kigami 1995].

For a graph tree T, one has

$$R_T(x,y) = d_T(x,y),$$

where d_T is the weighted shortest path metric, with edges weighted according to $(1/\lambda_e^G)_{e \in E(G)}$.

RESISTANCE CHARACTERISATION: REAL TREES

Again, let $(\mathcal{T}, d_{\mathcal{T}}, \rho_{\mathcal{T}})$ be a compact, rooted real tree, and $\mu^{\mathcal{T}}$ a finite Borel measure on $\mathcal{T}$ with full support.

Similarly to the graph case, define the resistance on $\mathcal{T}$ by

$$R_{\mathcal{T}}(x,y)^{-1} = \inf\left\{\mathcal{E}^{\mathcal{T}}(f,f) :\ f \in \mathcal{F}^{\mathcal{T}},\ f(x)=1,\ f(y)=0\right\}.$$

One can check that $R_{\mathcal{T}} = d_{\mathcal{T}}$. By results of [Kigami 1995] on 'resistance forms', it is possible to check that this property characterises $(\mathcal{E}^{\mathcal{T}}, \mathcal{F}^{\mathcal{T}})$ uniquely amongst the collection of regular Dirichlet forms on $L^2(\mu^{\mathcal{T}})$.

Note that, for all $f \in \mathcal{F}_{\mathcal{T}}$,

$$|f(x) - f(y)|^2 \leq \mathcal{E}_{\mathcal{T}}(f,f) d_{\mathcal{T}}(x,y).$$

PROOF OF POINT RECURRENCE

[Fukushima/Oshima/Takeda 2011, Lemma 2.2.3] If ν is a positive Radon measure on $\mathcal{T}$ with finite energy integral, i.e.,

$$\left(\int_{\mathcal{T}}|f(x)|\nu(dx)\right)^2 \leq c\left(\mathcal{E}^{\mathcal{T}}(f,f)+\int_{\mathcal{T}}f(x)^2\mu^{\mathcal{T}}(dx)\right), \quad \forall f\in\mathcal{F}^{\mathcal{T}},$$

then ν charges no set of zero capacity.

Note that

$$\left(\int_{\mathcal{T}}|f(z)|\delta_x(dz)\right)^2 = f(x)^2 \leq 2(f(x)-f(y))^2+2f(y)^2.$$

Applying the resistance inequality to this bound, and integrating with respect to y yields

$$\left(\int_{\mathcal{T}}|f(y)|\delta_x(dy)\right)^2 \leq 2\,\mathrm{diam}\mathcal{T}_f\ \mathcal{E}^{\mathcal{T}}(f,f)+2\int_{\mathcal{T}}f(y)^2\mu^{\mathcal{T}}(dy).$$

Thus points have strictly positive capacity.

PROOF OF OCCUPATION DENSITY FORMULA

Let $g(z) = g^y(x,z) = d_{\mathcal{T}}(b_{\mathcal{T}}(x,y,z),y)$, then

$$\nabla g = \mathbf{1}_{[[b_{\mathcal{T}}(\rho_{\mathcal{T}},x,y),x]]}(z) - \mathbf{1}_{[[b_{\mathcal{T}}(\rho_{\mathcal{T}},x,y),y]]}(z).$$

And for $h \in \mathcal{F}_{\mathcal{T}}$ with $h(y) = 0$,

$$\mathcal{E}_{\mathcal{T}}(g,h) = \int_{b_{\mathcal{T}}(\rho_{\mathcal{T}},x,y)}^{x} \nabla h(z)\lambda^{\mathcal{T}}(dz) - \int_{b_{\mathcal{T}}(\rho_{\mathcal{T}},x,y)}^{y} \nabla h(z)\lambda^{\mathcal{T}}(dz) = h(x).$$

Hence, if $Gf(x) := \int_{\mathcal{T}} g^y(x,z)f(z)\mu^{\mathcal{T}}(dz)$, then

$$\mathcal{E}_{\mathcal{T}}(Gf,h) = \int_{\mathcal{T}} f(z)h(z)\mu^{\mathcal{T}}(dz).$$

Since the resolvent is unique, to complete the proof it is enough to note that

$$\tilde{G}f(x) := E_x^{\mathcal{T}} \int_0^{\tau_y} f(X_s^{\mathcal{T}})ds = \int_0^{\infty} P_t^{\mathcal{T}\backslash\{y\}} f(x)dt$$

also satisfies the previous identity.

4. TRACES AND TIME CHANGE

TRACE OF THE DIRICHLET FORM

Through this section, let $(\mathcal{T}, d_{\mathcal{T}}, \rho_{\mathcal{T}})$ be a compact, rooted real tree, and $\mu^{\mathcal{T}}$ a finite Borel measure on $\mathcal{T}$ with full support.

Suppose $\mathcal{T}'$ is a non-empty subset of $\mathcal{T}$.

Define the trace of $(\mathcal{E}^{\mathcal{T}}, \mathcal{F}^{\mathcal{T}})$ on $\mathcal{T}'$ by setting:

$$\mathrm{Tr}\left(\mathcal{E}^{\mathcal{T}}|\mathcal{T}'\right)(g,g) := \inf\left\{\mathcal{E}^{\mathcal{T}}(f,f) : f \in \mathcal{F}^{\mathcal{T}},\ f|_{\mathcal{T}'} = g\right\},$$

where the domain of $\mathrm{Tr}(\mathcal{E}^{\mathcal{T}}|\mathcal{T}')$ is precisely the collection of functions for which the right-hand side is finite.

Theorem. If $\mathcal{T}'$ is closed, and $\mu^{\mathcal{T}'}$ is a finite Borel measure on $(\mathcal{T}', d_{\mathcal{T}})$ with full support, then $\mathrm{Tr}\left(\mathcal{E}^{\mathcal{T}}|\mathcal{T}'\right)$ is a regular Dirichlet form on $L^2(\mu^{\mathcal{T}'})$ [Fukushima/Oshima/Takeda 2011].

APPLICATION TO REAL TREES

Suppose $\mathcal{T}' \subseteq \mathcal{T}$ is closed and arcwise-connected (so that $(\mathcal{T}', d_{\mathcal{T}})$ is a real tree), equipped with a finite Borel measure $\mu^{\mathcal{T}'}$ of full support. We claim that

$$\mathcal{E}^{\mathcal{T}'} = \mathrm{Tr}\left(\mathcal{E}^{\mathcal{T}}|\mathcal{T}'\right).$$

Indeed, both are regular Dirichlet forms on $L^2(\mu^{\mathcal{T}'})$, and

$$\begin{aligned}
&\inf\left\{\mathrm{Tr}\left(\mathcal{E}^{\mathcal{T}}|\mathcal{T}'\right)(g,g):\ g(x)=1,\ g(y)=0\right\}\\
&= \inf\left\{\inf\left\{\mathcal{E}^{\mathcal{T}}(f,f):\ f\in\mathcal{F}^{\mathcal{T}},\ f|_{\mathcal{T}'}=g\right\}:\ g(x)=1,\ g(y)=0\right\}\\
&= \inf\left\{\mathcal{E}^{\mathcal{T}}(f,f):\ f\in\mathcal{F}^{\mathcal{T}},\ f(x)=1,\ f(y)=0\right\}\\
&= d_{\mathcal{T}}(x,y)^{-1}.
\end{aligned}$$

In particular, $\mathrm{Tr}(\mathcal{E}^{\mathcal{T}}|\mathcal{T}')$ is the form naturally associated with Brownian motion on $(\mathcal{T}', d_{\mathcal{T}}, \mu^{\mathcal{T}'})$.

TIME CHANGE

Given a finite Borel measure ν with support $\mathcal{S} \subseteq \mathcal{T}$, let $(A_t)_{t\geq 0}$ be the positive continuous additive functional with Revuz measure ν. For example, if $X^{\mathcal{T}}$ admits jointly continuous local times $(L_t(x))_{x\in\mathcal{T},t\geq 0}$, i.e.

$$\int_0^t f(X_s^{\mathcal{T}})ds = \int_{\mathcal{T}} f(x)L_t(x)\mu_{\mathcal{T}}(dx), \qquad \forall f \in C(\mathcal{T}),$$

then

$$A_t = \int_{\mathcal{S}} L_t(x)\nu(dx).$$

Set

$$\tau(t) := \inf\{s > 0 :\ A_s > t\}.$$

Then $(X^{\mathcal{T}}_{\tau(t)})_{t\geq 0}$ is the Markov process naturally associated with $\mathrm{Tr}\left(\mathcal{E}^{\mathcal{T}}|\mathcal{S}\right)$, considered as a regular Dirichlet form on $L^2(\nu)$.

APPLICATION TO FINITE SUBSETS

Let V be a fine finite set of $\mathcal{T}$. If we define $\mathcal{E}^V = \mathrm{Tr}(\mathcal{E}^{\mathcal{T}}|V)$, then one can check for any finite measure μ^V on V with full support

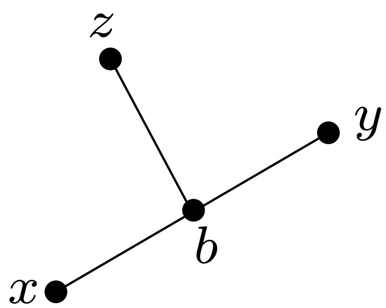
$$\begin{aligned} \mathcal{E}^V(f,g) &= \frac{1}{2} \sum_{x,y:x\sim y} \frac{1}{d_{\mathcal{T}}(x,y)} \left(f(x)-f(y)\right)\left(g(x)-g(y)\right) \\ &= -\sum_x (\Delta f)(x) g(x) \mu^V(\{x\}), \end{aligned}$$

where

$$\Delta f(x) := \sum_{y:y\sim x} \frac{1}{\mu^V(\{x\}) d_{\mathcal{T}}(x,y)} \left(f(y)-f(x)\right).$$

PROOF OF HITTING PROBABILITIES FORMULA

Let $V=\{x,y,z,b_{\mathcal{T}}(x,y,z)\}$.



For any μ^V such that $\mu(\{v\})\in(0,\infty)$ for all $v\in V$, we have $P_x^{\mathcal{T}}$-a.s.,

$$A_t=\int_0^t \mathbf{1}_V(X_s^{\mathcal{T}})dA_s, \qquad \inf\{t:A_t>0\}=\inf\{t:X_t^{\mathcal{T}}\in V\}.$$

[Fukushima/Oshima/Takeda 2011] It follows that the hitting distributions of $X_t^V=X_{\tau(t)}^{\mathcal{T}}$ and $X^{\mathcal{T}}$ are the same. Thus

$$P_z^{\mathcal{T}}\left(\tau_x<\tau_y\right)=P_z^V\left(\tau_x<\tau_y\right)=\frac{d_{\mathcal{T}}(b_{\mathcal{T}}(x,y,z),y)}{d_{\mathcal{T}}(x,y)}.$$

5. SCALING RANDOM WALKS ON GRAPH TREES

AIM

Let $(T_n)_{n\geq 1}$ be a sequence of finite graph trees, and μ_{T_n} the counting measure on $V(T_n)$.

(A1) There exist null sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ such that

$$\left(T_n, a_n d_{T_n}, b_n \mu_{T_n}, \rho_{T_n}\right) \to (\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \rho_{\mathcal{T}})$$

with respect to the pointed Gromov-Hausdorff-Prohorov topology.

We aim to show that the corresponding simple random walks X^{T_n}, started from ρ_{T_n}, converge to Brownian motion $X^{\mathcal{T}}$ on $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}})$, started from $\rho_{\mathcal{T}}$.

ASSUMPTION ON LIMIT

From the convergence assumption (A1) we have that: $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \rho_{\mathcal{T}})$ is a compact real tree, equipped with a finite Borel measure $\mu^{\mathcal{T}}$, and distinguished point $\rho_{\mathcal{T}}$.

(A2) There exists a constant $c > 0$ such that

$$\liminf_{r \to 0} \inf_{x \in \mathcal{T}} r^{-c} \mu_{\mathcal{T}}\left(B_{\mathcal{T}}(x, r)\right) > 0.$$

This property is not necessary, but allows a sample path proof.

In particular, it ensures that $X^{\mathcal{T}}$ admits jointly continuous local times $(L_t(x))_{x \in \mathcal{T}, t \geq 0}$, i.e.

$$\int_0^t f(X_s^{\mathcal{T}}) ds = \int_{\mathcal{T}} f(x) L_t(x) \mu_{\mathcal{T}}(dx), \qquad \forall f \in C(\mathcal{T}).$$

A NOTE ON THE TOPOLOGY

The assumption (A1) is equivalent to there existing isometric embeddings of $(T_n, d_{T_n})_{n\geq 1}$ and $(\mathcal{T}, a_n d_{\mathcal{T}})$ into the same metric space (M, d_M) such that:

$$d_M(\rho_{T_n}, \rho_{\mathcal{T}}) \to 0, \qquad d_M^H(T_n, \mathcal{T}) \to 0, \qquad d_M^P(b_n \mu_{T_n}, \mu_{\mathcal{T}}) \to 0.$$

Indeed, one can take

$$M = T_1 \sqcup T_2 \sqcup \cdots \sqcup \mathcal{T}$$

equipped with suitable metric (cf. end of Section 2).

We will identify the various objects with their embeddings into M, and show convergence of processes in the space $D(\mathbb{R}_+, M)$.

PROJECTIONS

Let $(x_i)_{i\geq 1}$ be a dense sequence in $\mathcal{T}$, and set

$$\mathcal{T}(k) := \cup_{i=1}^k [[\rho_{\mathcal{T}}, x_i]],$$

where $[[\rho_{\mathcal{T}}, x_i]]$ is the unique path from $\rho_{\mathcal{T}}$ to x_i in $\mathcal{T}$.

Let $\phi_k : \mathcal{T} \to \mathcal{T}(k)$ be the map such that $\phi_k(x)$ is the nearest point of $\mathcal{T}(k)$ to x. (We call this the **projection** of $\mathcal{T}$ onto $\mathcal{T}(k)$.)

For each n, choose $(x_i^n)_{i\geq 1}$ in T_n such that

$$d_M(x_i^n, x_i) \to 0,$$

and define the subtree $T_n(k)$ and projection $\phi_{n,k} : T_n \to T_n(k)$ similarly to above.

CONVERGENCE CRITERIA

It is possible to check that the assumption (A1) is equivalent to the following two conditions holding:

1. Convergence of finite dimensional distributions: for each k,

$$d_M^H(T_n(k), \mathcal{T}(k)) \to 0, \qquad d_M^P(b_n \mu_{n,k}, \mu_k) \to 0,$$

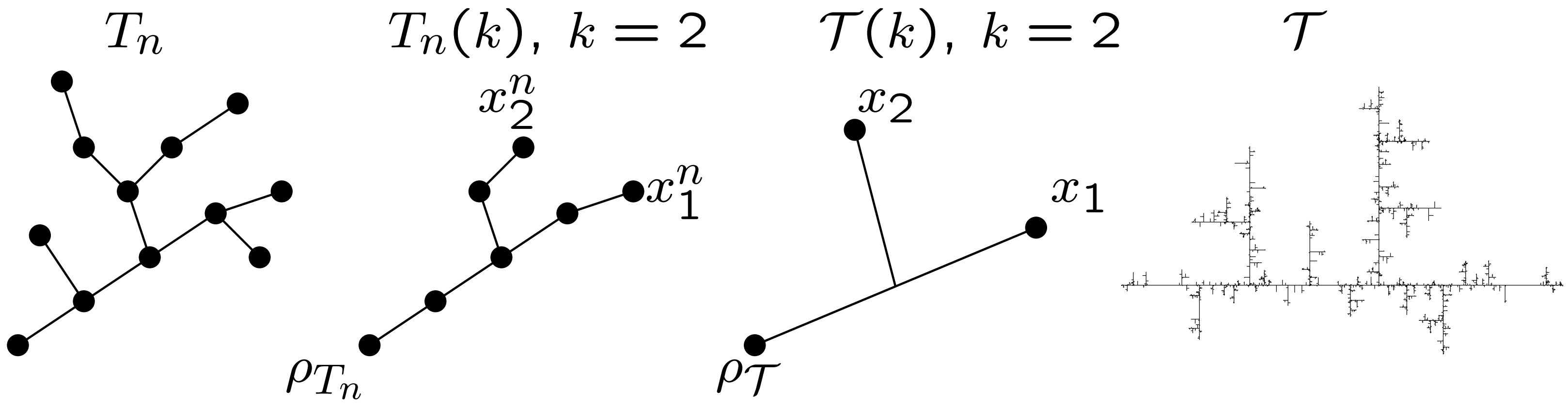
where $\mu_{n,k} := \mu_{T_n} \circ \phi_{n,k}^{-1}$ and $\mu_k := \mu_{\mathcal{T}} \circ \phi_k^{-1}$.

2. Tightness:

$$\lim_{k \to \infty} \limsup_{n \to \infty} d_M^H(T_n(k), T_n) = 0.$$

STRATEGY

Select $T_n(k)$ and $\mathcal{T}(k)$ as above:



Step 1: Show Brownian motion $X^{\mathcal{T}(k)}$ on $(\mathcal{T}(k), d_{\mathcal{T}}, \mu_k)$ converges to $X^{\mathcal{T}}$.
Step 2: For each k, construct processes $X^{T_n(k)}$ on graph subtrees that converge to $X^{\mathcal{T}(k)}$.
Step 3: Show $X^{T_n(k)}$ are close to X^{T_n} as $k \to \infty$.

STEP 1
APPROXIMATION OF LIMITING DIFFUSION

TIME CHANGE CONSTRUCTION

Define

$$A_t^k := \int_{\mathcal{T}} L_t(x)\mu_k(dx),$$

set

$$\tau_k(t) = \text{inf}\{s : \ A_s^k > t\}.$$

Then, we recall from Section 4, $X^{\mathcal{T}}_{\tau_k(t)}$ is the Markov process naturally associated with

$$\text{Tr}\left(\mathcal{E}^{\mathcal{T}}|\mathcal{T}(k)\right),$$

(note that supp$\mu_k = \mathcal{T}(k)$), considered as a Dirichlet form on $L^2(\mu_k)$.

Recall also that the latter process is Brownian motion $X^{\mathcal{T}(k)}$ on $(\mathcal{T}(k), d_{\mathcal{T}}, \mu_k)$.

CONVERGENCE OF DIFFUSIONS

By construction

$$d_M^P(\mu_k, \mu_{\mathcal{T}}) \leq \sup_{x \in \mathcal{T}} d_M(\phi_k(x), x) = d_M^H(\mathcal{T}(k), \mathcal{T}) \to 0.$$

Hence, applying the continuity of local times:

$$A_t^k = \int_{\mathcal{T}} L_t(x) \mu_k(dx) \to \int_{\mathcal{T}} L_t(x) \mu_{\mathcal{T}}(dx) = t,$$

uniformly over compact intervals.

Thus, we also have that $\tau_k(t) \to t$ uniformly on compacts. And, by continuity,

$$X_t^{\mathcal{T}(k)} = X_{\tau_k(t)}^{\mathcal{T}} \to X_t^{\mathcal{T}},$$

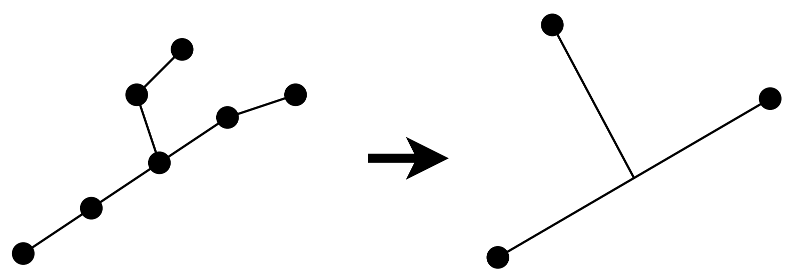
uniformly on compacts.

STEP 2
CONVERGENCE OF WALKS ON FINITE TREES

CONVERGENCE OF WALKS ON FINITE TREES EQUIPPED WITH LENGTH MEASURE

For fixed k,

$$T_n(k) \to \mathcal{T}(k).$$



If $J^{n,k}$ is the simple random walk on $T_n(k)$, then

$$\left(J^{n,k}_{tE_{n,k}/a_n}\right)_{t\geq 0} \to \left(X_t^{\mathcal{T}(k),\lambda_k}\right)_{t\geq 0},$$

where $E_{n,k} := \#E(T_n(k))$ and $X^{\mathcal{T}(k),\lambda_k}$ is the Brownian motion on $(\mathcal{T}(k), d_{\mathcal{T}}, \lambda_k)$, for λ_k equal to the length measure on $\mathcal{T}(k)$, normalised such that $\lambda_k(\mathcal{T}(k)) = 1$.

TIME CHANGE FOR LIMIT

For $(L^k_t(x))_{x\in\mathcal{T}(k),t\geq 0}$ the local times of $X^{\mathcal{T}(k),\lambda_k}$, write

$$\hat{A}^k_t := \int_{\mathcal{T}(k)} L^k_t(x)\mu_k(dx),$$

and set

$$\hat{\tau}_k(t) = \inf\{s:\ \hat{A}^k_s > t\}.$$

Then

$$\left(X^{\mathcal{T}(k),\lambda_k}_{\hat{\tau}_k(t)}\right)_{t\geq 0} = \left(X^{\mathcal{T}(k)}_t\right)_{t\geq 0}.$$

TIME CHANGE FOR GRAPHS

Let

$$\widehat{A}_m^{n,k} := \sum_{l=0}^{m-1} \frac{2\mu_{n,k}(\{J_l^{n,k}\})}{\mathrm{deg}_{n,k}(J_l^{n,k})} = \sum_{x \in T_n(k)} L_m^{n,k}(x)\mu_{n,k}(\{x\}),$$

where

$$L_m^{n,k}(x) := \frac{2}{\mathrm{deg}_{n,k}(x)} \sum_{l=0}^{m-1} \mathbf{1}_{\{J_l^{n,k}=x\}}.$$

If

$$\widehat{\tau}^{n,k}(m) := \max\{l : \widehat{A}_l^{n,k} \leq m\},$$

then

$$X_m^{T_n(k)} = J_{\widehat{\tau}^{n,k}(m)}^{n,k}$$

is the process with the same jump chain as $J^{n,k}$, and holding times given by $2\mu_{n,k}(\{x\})/\mathrm{deg}_{n,k}(x)$.

CONVERGENCE OF TIME-CHANGED PROCESSES

We have that

$$\left(a_n L^{n,k}_{tE_{n,k}/a_n}(x)\right)_{x\in T_n(k),t\geq 0} \to \left(L^k_t(x)\right)_{x\in\mathcal{T}(k),t\geq 0}, \qquad b_n\mu_{n,k}\to\mu_k.$$

This implies which implies

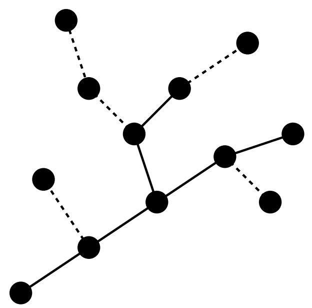
$$\begin{aligned} a_n b_n \widehat{A}^{n,k}_{tE_{n,k}/a_n} &= a_n b_n \int_{T_n(k)} L^{n,k}_{tE_{n,k}/a_n}(x)\mu_{n,k}(dx) \\ &\to \int_{\mathcal{T}(k)} L^k_t(x)\mu_k(dx) \\ &= \widehat{A}^k_t. \end{aligned}$$

Taking inverses and composing with $J^{n,k}$ and $X^{\mathcal{T}(k),\lambda_k}$ yields

$$X^{T_n(k)}_{t/a_nb_n} = J^{n,k}_{\widehat{\tau}^{n,k}(t/a_nb_n)} \to X^{\mathcal{T}(k),\lambda_k}_{\widehat{\tau}_k(t)} = X^{\mathcal{T}(k)}_t.$$

STEP 3
APPROXIMATING RANDOM WALKS ON WHOLE TREES

PROJECTION OF RANDOM WALK



$\phi_{n,k}$ is natural projection from T_n to $T_n(k)$.

Clearly

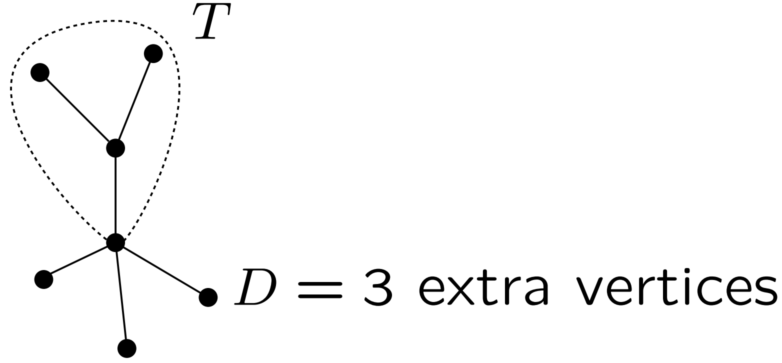
$$\begin{aligned}
&\lim_{k\to\infty} \limsup_{n\to\infty} \sup_{t\in[0,T]} d_M\left(X^{T_n}_{t/a_nb_n}, \phi_{n,k}(X^{T_n}_{t/a_nb_n})\right) \\
&\leq \lim_{k\to\infty} \limsup_{n\to\infty} \sup_{x\in V(T_n)} d_M(x, \phi_{n,k}(x)) \\
&= \lim_{k\to\infty} \limsup_{n\to\infty} d_M^H(T_n(k), T_n) \\
&= 0.
\end{aligned}$$

Moreover, can couple projected process $\phi_{n,k}(X^{T_n})$ and time-changed process $X^{T_n(k)}$ to have same jump chain $J^{n,k}$. Recall $X^{T_n(k)}$ waits at a vertex x a fixed time $2\mu_{n,k}(\{x\})/\text{deg}_{n,k}(x)$.

ELEMENTARY SIMPLE RANDOM WALK IDENTITY

Let T be a rooted graph tree, and attach D extra vertices at its root, each by a single edge.

e.g.



If $\alpha(T, D)$ is the expected time for a simple random walk started from the root to hit one of the extra vertices, then

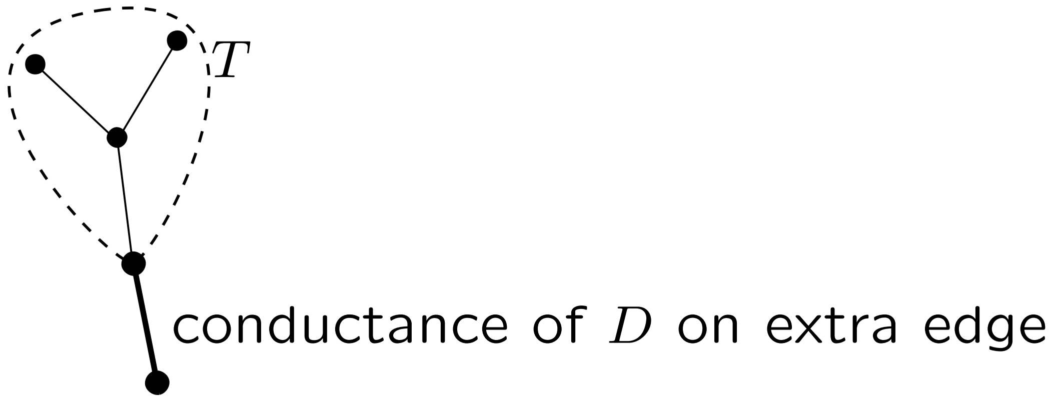
$$\alpha(T, D) = \frac{2\#V(T) - 2 + D}{D}.$$

In particular, if $D = 2$, then

$$\alpha(T, D) = \#V(T).$$

PROOF

We consider modified graph $G = T \cup \{\rho\}$ obtained by identifying extra vertices into one vertex:



If τ_ρ^+ is the return time to ρ, then

$$\alpha(T, D) + 1 = E_\rho^G \tau_\rho^+ = \frac{1}{\pi(\rho)},$$

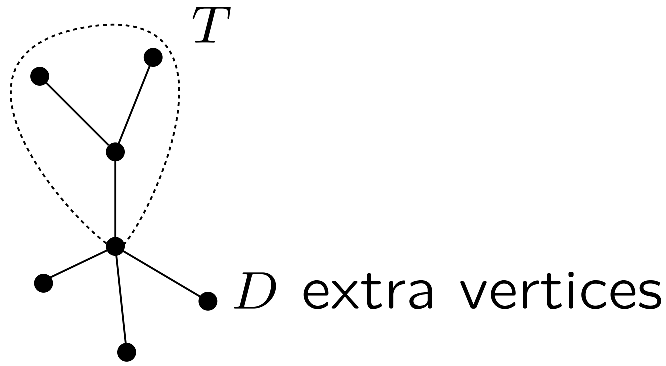
where π is the invariant probability measure of the random walk. In particular, writing $\lambda(v) = \sum_{e:\, v \in e} \lambda_e$,

$$\pi(\rho) = \frac{\lambda(\rho)}{\sum_v \lambda(v)} = \frac{D}{2(D + \#E(T))} = \frac{D}{2(D + \#V(T) - 1)}.$$

SECOND MOMENT ESTIMATE

Again, let T be a rooted graph tree, and attach D extra vertices at its root, each by a single edge.

e.g.



If $\beta(T,D)$ is the second moment of the time for a simple random walk started from the root to hit one of the extra vertices, then there exists a universal constant c such that

$$\beta(T,D) \leq c\left(\#V(T)^2 \times (1+h(T)) + Dh(T)\right),$$

where $h(T)$ is the height of T.

PROOF

Let $G = T \cup \{\rho\}$ be the modified graph as in the previous proof. If $\lambda(G) = \sum_v \lambda(v) = 2\sum_e \lambda_e$ and $r(G) = \max_{x,y \in G} R(x,y)$, then we claim

$$P_\rho^G\left(\tau_\rho^+ \geq a\right) \leq \frac{c_1}{r(G)D} e^{-c_2 a/\lambda(G) r(G)}.$$

Indeed, applying the Markov property repeatedly, we obtain

$$P_\rho^G\left(\tau_\rho^+ \geq a\right) \leq P_\rho^G\left(\tau_\rho^+ \geq a/k\right)\left(\max_{x \in V(T)} P_x^G\left(\tau_\rho \geq a/k\right)\right)^{k-1}.$$

For $k = a/2\lambda(G)r(G)$, we have

$$P_\rho^G\left(\tau_\rho^+ \geq a/k\right) \leq \frac{k E_\rho^G \tau_\rho^+}{a} = \frac{1}{2r(G)D},$$

and also, by the commute time identity,

$$\max_{x \in V(T)} P_x^G\left(\tau_x \geq a/k\right) \leq \max_{x \in V(T)} \frac{k E_x^G \tau_\rho}{a} \leq \max_{x \in V(T)} \frac{k R(x,\rho)\lambda(G)}{a} \leq \frac{1}{2}.$$

PROOF (CONT.)

It follows that

$$E_{\rho}^{G}\left((\tau_{\rho}^{+})^{2}\right) \leq \frac{c_3 \lambda(G)^2 r(G)}{D}.$$

Since

$$\beta(T,D) = E_{\rho}^{G}\left((\tau_{\rho}^{+}-1)^{2}\right),$$

we can then use that

$$\lambda(G) = 2(D + \#V(T) - 1), \qquad r(G) \leq 2(h(T) + D^{-1})$$

to complete the proof.

CLOSENESS OF CLOCK PROCESSES

Suppose the mth jump of $\phi_{n,k}(X^{T_n})$ happens at $A_m^{n,k}$. Applying the above moment estimates and Kolmogorov's maximum estimate, i.e. if X_i are independent, centred, then

$$\mathbf{P}(\max_{l=1,\ldots,m} |\sum_{i=1}^{l} X_i| \geq x) \leq x^{-2} \sum_{i=1}^{m} \mathbf{E} X_i^2,$$

we deduce

$$\mathbf{P}\left(\max_{m \leq t E_{n,k}/a_n} \left| A_m^{n,k} - \widehat{A}_m^{n,k} \right| \geq \varepsilon / a_n b_n \right) \to 0$$

in probability as n and then k diverge.

CONCLUSION

Let $(T_n)_{n\geq 1}$ be a sequence of finite graph trees.

Suppose that there exist null sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ such that

$$\left(T_n, a_n d_{T_n}, b_n \mu_{T_n}, \rho_{T_n}\right) \to (\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \rho_{\mathcal{T}})$$

with respect to the pointed Gromov-Hausdorff-Prohorov topology, and $\mathcal{T}$ satisfies a polynomial lower volume bound.

It is then possible to isometrically embed $(T_n)_{n\geq 1}$ and $\mathcal{T}$ into the same metric space (M, d_M) such that

$$\left(a_n X^{T_n}_{t/a_n b_n}\right)_{t\geq 0} \to \left(X^{\mathcal{T}}_t\right)_{t\geq 0}$$

in distribution in $C(\mathbb{R}_+, M)$, where we assume $X_0^{T_n} = \rho_{T_n}$ for each n, and also $X_0^{\mathcal{T}} = \rho_{\mathcal{T}}$.

REMARKS

(i) Can extend to locally compact case.
(ii) Alternative proof given in [Athreya/ Löhr/Winter 2014] (in a slightly more general setting) under the weaker assumption: for each $\delta > 0$,

$$\liminf_{n\to\infty} \inf_{x\in T_n} \mu_{T_n}(B_{T_n}(\rho_{T_n}, \delta/a_n)) > 0.$$

(iii) Embeddings can be described measurably, and chosen so result applies to random trees to give convergence of annealed laws. In particular, if

$$\left(T_n, a_n d_{T_n}, b_n \mu_{T_n}, \rho_{T_n}\right) \to (\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \rho_{\mathcal{T}})$$

in distribution, then for appropriate embeddings

$$\int P^{T_n}_{\rho_{T_n}}((a_n X^{T_n}_{t/a_n b_n})_{t\geq 0} \in \cdot)\mathbb{P}(dT_n) \to \int P^{\mathcal{T}}_{\rho_{\mathcal{T}}}((X^{\mathcal{T}}_t)_{t\geq 0} \in \cdot)\mathbb{P}(d\mathcal{T}).$$

Applies to critical, finite variance GW trees conditioned on their size, with $a_n = n^{-1/2}$, $b_n = n^{-1}$.